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Research Article

A Rational Study on Limit States in Hypoelastic Materials

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ABSTRACT

Despite the definition of failure being almost thoroughly studied in other theories such as elasto-plasticity, this paper studies the possibility of capturing or defining some limit states in hypoelastic materials. It is shown that for many hypoelastic materials the limit state, as a notion of failure, can take the place of the yield or failure in classical plasticity. The procedure is general, and all equations are rational, i.e. they are not dependent on a particular form of a constitutive equation. Constitutive equations cover those applied to both metallic and non-metallic materials. Some practical results were obtained for a particular form of a hypoelastic equation which resembles the Drucker-Prager criterion for the form of the limit state. Results were also examined against a set of experimental data.

1. Introduction

In the context of the classical plasticity theory, the yield is often defined as the boundary separating the elastic region from the inelastic one. In one dimension, in particular in metals, the yield point may be very sharp (e.g. in mild steel) or smooth (e.g. in aluminum). This is also the case in granular materials [1, 2]. Although the definition of yield is often accurate, the definition of failure (or rupture) is, to some extent, arbitrary. In contrast to perfectly plastic materials, where the yield and failure are identical, in hardening materials, there is no accurate definition of failure. In general, the failure is highly dependent on the interpretation of the examiner [3, 4]. Locally, it sometimes means the stress tensor remains stationary, and hence, its material derivative vanishes. This is called the limit state in the context of solid mechanics and is widely used in geomechanics (e.g. [5-7]).

Even though in the theory of plasticity, a system of

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independent equations is required in cases such as that of elastic behavior, yield criterion, hardening, and flow rule among others, in the fairly novel theory of hypoelasticity, there is only one constitutive equation governing the material behavior. This theory, originally developed in the mid-1900s, has been successfully developed by both hypoelasticity and hypoplasticity, to predict the behavior of solids including metals and geomaterials [8-16].

It has been long a question whether the hypoelastic constitutive equations, or in general, hypo-materials, can predict or capture a limit state. Research in this area [10, 16, 17-19] is to some extent very limited making it an open research area.

In this research, the ability of hypoelastic constitutive equations in terms of predicting the limit state is studied. The form of the hypoelastic equation is arbitrary but examples are provided for some simple forms. In addition, the use of an objective stress rate in the definition of the hypoelastic constitutive equation has been made and results can be applied to finite (also called large) deformation analysis, widely applied in practice [20]. The novelty of the work may fall into presenting a simple hypoelastic constitutive equation which is equivalent to elasto-plastic models also approaching a limit state with a form corresponding to the Drucker-Prager criterion. In addition, two possible alternatives to define the limit state were presented and compared.

2. Theory of Hypoelastic Materials

A hypoelastic material is a sub-class of materials of the rate type [21]. This is defined as one with a linear functional dependence of some objective stress rate on the rate of deformation tensor [9]. The general form of the hypoelastic constitutive equation is as follows:

$$\boldsymbol{T}^{\blacksquare} = \boldsymbol{h}(\boldsymbol{T},\boldsymbol{D},\rho) = \mathbb{C}^{he}\!:\boldsymbol{D}, \qquad \mathbb{C}^{he} = \mathbb{C}^{he}(\boldsymbol{T},\rho) \qquad (1)$$

where $\boldsymbol{T}^{\blacksquare}$ is an objective stress rate, $\boldsymbol{h}$ is the hypoelasticity function, $\boldsymbol{T}$ is the stress tensor, $\boldsymbol{D}$ is the rate of deformation tensor, ρ is the density and $\mathbb{C}^{he}$ is the 4th order hypoelastic constitutive tensor.

The most general form of the hypoelastic constitutive equation with b_i's being functions of the invariants of the stress tensor is as follows [9]:

$$\boldsymbol{T}^{\blacksquare} = b_1\boldsymbol{D} + b_2(\boldsymbol{TD}+\boldsymbol{DT}) + b_3(\boldsymbol{T}^2\boldsymbol{D}+\boldsymbol{DT}^2) + [b_4\mathrm{tr}\boldsymbol{D} + b_5\mathrm{tr}(\boldsymbol{TD}) + b_6\mathrm{tr}(\boldsymbol{T}^2\boldsymbol{D})]\boldsymbol{I} + [b_7\mathrm{tr}\boldsymbol{D} + b_8\mathrm{tr}(\boldsymbol{TD}) + b_9\mathrm{tr}(\boldsymbol{T}^2\boldsymbol{D})]\boldsymbol{T} + [b_{10}\mathrm{tr}\boldsymbol{D} + b_{11}\mathrm{tr}(\boldsymbol{TD}) + b_{12}\mathrm{tr}(\boldsymbol{T}^2\boldsymbol{D})]\boldsymbol{T}^2 \qquad (2)$$

This equation can take simpler forms by physical constraints and requirements stemmed from physical observations. For instance, by keeping only b_1 and b_4 and letting the rest of constants vanish, the following form is reduced to the generalized Hooke's law in linear elasticity by application of infinitesimal deformation:

$$\boldsymbol{T}^{\blacksquare} = b_4(\mathrm{tr}\boldsymbol{D})\boldsymbol{I} + b_1\boldsymbol{D} = \lambda(\mathrm{tr}\boldsymbol{D})\boldsymbol{I} + 2\mu\boldsymbol{D} \qquad (3)$$

where $\lambda = b_4$ and $2\mu = b_1$ are Lamé constants.

If one makes the approximation $\boldsymbol{D} \cong \dot{\boldsymbol{\varepsilon}}$ for infinitesimal deformations and hence, $\boldsymbol{T}^{\blacksquare} \cong \dot{\boldsymbol{T}}$, the equation will be exactly the generalized form of Hooke's law in the rate form. This equation is linear but may exhibit a nonlinear material behavior. In addition, based on the chosen objective, stress rate in place of $\boldsymbol{T}^{\blacksquare}$ different behaviors may be observed, e.g. an oscillatory behavior for the famous Jaumann-Zaremba-Noll (Jaumann) stress rate. Fig. 1 shows three different responses of such materials under a simple shear motion, with dilation, under the Jaumann, Oldroyd and Truesdell objective stress rates for $\lambda = \mu = 1$ and p_0 is the initial value of $\mathrm{tr}\boldsymbol{T}/3$. The oscillatory behavior does not necessitate a periodic behavior, other than in the case of pure shear deformation (with no dilation), as a special case, where the response is both oscillatory and periodic. A detailed discussion can be found in the literature (e.g. [1]).

One should note that the hypoelasticity equation can be regarded as a linear transformation (or operator) from the space of all 2nd order tensors into itself, i.e. $\boldsymbol{h}: S \to S$ where $S(= \mathbb{R}^3 \times \mathbb{R}^3)$ is the vector space corresponding to all [symmetric] 2nd order tensors. Therefore, $\mathbb{C}^{he}$ can be represented by the matrix of this transformation which will later be used.

3. The Limit State

Analogous to the definition of failure (or yield, in perfectly plastic materials) the limit state can be defined

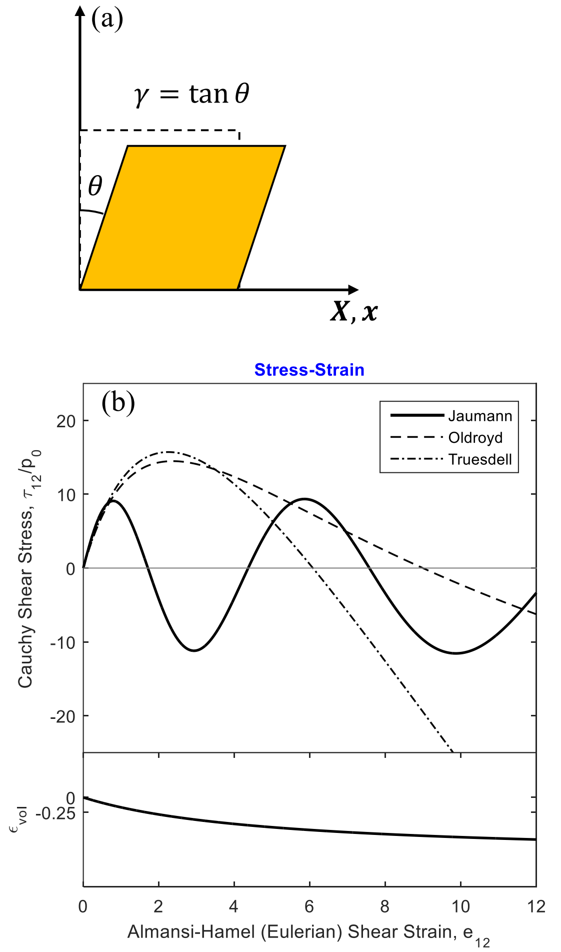

Fig. 1. Material response to the simple shear motion with different objective stress rates; (a) simple shear motion field, (b) stress-strain curve.

as the state in which the stress tensor becomes constant. It means the material derivative of $\boldsymbol{T}$ vanishes. This is the dynamical condition of the limit state. In addition, the kinematical condition is often prescribed to define the limit state. It defines it as a state where only shear deformation takes place and the volume remains constant. Thus, the kinematical condition is $\mathrm{tr}\boldsymbol{D} = \mathrm{div}\boldsymbol{v} = 0$.

The first condition is sometimes simplified as $\boldsymbol{T}^{\blacksquare} = \boldsymbol{0}$. However, $\dot{\boldsymbol{T}} = \boldsymbol{0}$ does not necessitate that $\boldsymbol{T}^{\blacksquare} = \boldsymbol{0}$. An important study conducted by [22] indicated that the assumption of $\boldsymbol{T}^{\blacksquare} = \boldsymbol{0}$ often leads to $\dot{\boldsymbol{T}} = \boldsymbol{0}$ for a series of hypoplastic constitutive equations, although it is still debatable. This may also be true for hypoelastic materials, with simpler constitutive equations. It seems that within the practical range of deformations under infinitesimal or no vorticity (e.g. in proportional loading) at the limit state, the difference between $\boldsymbol{T}^{\blacksquare}$ and $\dot{\boldsymbol{T}}$ is negligible, assuming $\boldsymbol{T}^{\blacksquare} = \boldsymbol{0}$ valid for practical purposes. Therefore, one approach to check whether the limit state can be reached is to check whether $\dot{\boldsymbol{T}} = \boldsymbol{0}$ may give a meaningful equation for the stress tensor. The second condition, i.e. the kinematical condition, depends on experimental observations and may or may not be applicable. One should note that a meaningful equation is one where the stress tensor, $\boldsymbol{T}$, satisfies rigor constraints, i.e. it is real (all eigenvalues or principal stresses are real) and delineates a well-defined boundary in the stress space with clearly observable inside and outside regions. For such a limit state, the stress paths can be traced to see if they eventually approach or reach this boundary, noting that, stresses beyond it (or outside this region) will be impossible to reach. One unnecessary condition can be the convexity of the limit state surface which is required by the classical theory of plasticity for the purpose of creating consistency with conventional yield/failure surfaces.

Now, the question is that under what condition(s) may $\boldsymbol{T}^{\blacksquare}$ (or $\dot{\boldsymbol{T}}$) vanish? An immediate answer is that the vanishing of $\boldsymbol{T}^{\blacksquare}$, for example, requires the vanishing of $\mathbb{C}^{he}$. However, in the context of linear algebra, assuming that the form of the hypoelastic constitutive equation is a linear operator, the vanishing of $\boldsymbol{T}^{\blacksquare}$ corresponds to a homogeneous system of linear algebraic equations in $\boldsymbol{T}$. Therefore, $\boldsymbol{T}^{\blacksquare} = \mathbb{C}^{he}:\boldsymbol{D} = \boldsymbol{0}$ corresponds to conditions on $\mathbb{C}^{he}$ such that the kernel of the linear operator becomes non-empty. In other words, the question will be under what conditions on $\boldsymbol{T}$ will the kernel of $\mathbb{C}^{he}$ become non-empty? This is equivalent to saying that under some stress states, the stress rate vanishes for a set of motions ($\boldsymbol{D}$). This approach was also put forth by [10] and here, we try to extend it in order to study the forms of the limit states obtained by the hypoelastic constitutive equations. To do so, we first examine and compare the two different approaches for a linear hypoelastic constitutive equation. Since the approach is general and does not depend on any specific form of a constitutive equation, we employ a simple form of the constitutive equation given below:

$$\boldsymbol{T}^{\blacksquare} = c_0(\mathrm{tr}\boldsymbol{T})(\mathrm{tr}\boldsymbol{D})\boldsymbol{I} + c_1(\mathrm{tr}\boldsymbol{T})\boldsymbol{D} + \frac{c_2\mathrm{tr}(\boldsymbol{TD})\boldsymbol{T}}{\mathrm{tr}\boldsymbol{T}} \quad (4)$$

The reason for this choice is that the first two terms correspond to the famous equation of the generalized Hooke's law, with a slight difference. The difference lies in the constants which are now functions of the isotropic stress, i.e. $\lambda = c_0(\mathrm{tr}\boldsymbol{T})$ and $2\mu = c_1(\mathrm{tr}\boldsymbol{T})$. This dependence is advantageous in porous and granular materials where the behavior is pressure-dependent and the stiffness increases with the confining pressure. The third term contains $\mathrm{tr}(\boldsymbol{TD})$ which is a measure of the rate of the stress work. In the theory of plasticity, this quantity often serves as a measure of hardening and the third term can be assumed to be a decaying term causing the material stiffness to decrease. In one-dimension cases, it causes the stress to gradually decrease in a monotonically increasing deformation (but in higher dimensions, increase or decrease for the stress tensor is meaningless, hence, leading to the use of the word stiffness to convey the meaning). In the present work, the necessity of the limit state by both the first and second approaches is investigated and discussed.

4. The Limit State by the First Approach

In the first approach, we require the constitutive tensor, $\mathbb{C}^{he}$, to vanish. The dynamical condition, which is mandatory, as well as the kinematical condition, which is arbitrary, are studied.

4.1. Dynamical condition

$$\boldsymbol{T}^{\blacksquare} = \mathbb{C}^{he}:\boldsymbol{D} = \boldsymbol{0},\ \boldsymbol{D} \neq \boldsymbol{0} \ \text{ requires } \ \mathbb{C}^{he} = \boldsymbol{0} \quad (5a)$$

$$\mathbb{C}^{he}_{ijkl} = \left[c_0 t_{mm}\delta_{ij}\delta_{kl} + c_1 t_{mm}\delta_{ik}\delta_{jl} + \frac{c_2 t_{ij}t_{kl}}{t_{mm}}\right] = 0 \quad (5b)$$

In these equations, t_{ij} are components of the stress tensor, δ_{ij} are components of the isotropic tensor of the 2nd order which is often called the Kronecker delta and $\mathbb{C}^{he}_{ijkl}$ are components of the 4th order hypoelastic constitutive tensor. Thus:

$$\left(c_0\delta_{ij}\delta_{kl} + c_1\delta_{ik}\delta_{jl}\right)t_{mm} + \frac{c_2 t_{ij}t_{kl}}{t_{mm}} = 0 \quad (5c)$$

By making two contractions on $i = k$ and $j = l$ the equation takes the form:

$$\left(c_0\delta_{ij}\delta_{ij} + c_1\delta_{ii}\delta_{jj}\right)t_{mm} + \frac{c_2 t_{ij}t_{ij}}{t_{mm}} = 0 \quad (5d)$$

Thus, the equation of the limit state under the sole dynamical condition is as follows:

$$(3c_0 + 9c_1)t_{mm} + \frac{c_2 t_{ij}t_{ij}}{t_{mm}} = 0 \quad (6a)$$

$$\mathrm{tr}\boldsymbol{T}^2 + c(\mathrm{tr}\boldsymbol{T})^2 = 0 \quad \text{or} \quad (1+c)I_1^2 - 2I_2 = 0, \qquad c = \frac{(3c_0 + 9c_1)}{c_2} \quad (6b)$$

$$J_2 + \kappa^2 I_1^2 = 0,\ \kappa^2 = -\frac{(1+3c)}{6},\ c < -\frac{1}{3} \quad (6c)$$

In these equations, I_1 and I_2 are the first and second invariants of the stress tensor, respectively, while J_2 is the second invariant of the deviatoric stress tensor.

The equation of the limit state surface corresponds to a circular cone in $\sigma_1 - \sigma_2 - \sigma_3$ stress space (or the Haigh-Westergaard space) which resembles the Drucker-Prager yield/failure surface with an elliptical cross-section in $\sigma_2 - \sigma_3$ plane. Fig. 2 shows the form of this limit state surface in its stress space and a cross-section. If the state of failure is known, e.g. from the experimental results, the remaining two constants, e.g. c_0 and c_1 can be found by making measurements on the elastic branches of the stress-strain curves. This is similar to how the elastic constants in a linear elastic material are determined. One should also note that this is a necessary condition for the limit state to be reached.

4.2. Kinematical condition

If one wishes to impose the second condition (the kinematical condition, i.e. $\mathrm{tr}\boldsymbol{D} = 0$), it requires:

$$\begin{aligned}\boldsymbol{T}^{\blacksquare} &= c_0(tr\boldsymbol{T})(tr\boldsymbol{D})\boldsymbol{I} + c_1(tr\boldsymbol{T})\boldsymbol{D} + \frac{c_2 tr(\boldsymbol{TD})\boldsymbol{T}}{tr\boldsymbol{T}} \\ &= 0 + c_1(\mathrm{tr}\boldsymbol{T})\boldsymbol{D} + \frac{c_2\mathrm{tr}(\boldsymbol{TD})\boldsymbol{T}}{\mathrm{tr}\boldsymbol{T}} = \boldsymbol{0}\end{aligned} \quad (7a)$$

Thus:

$$c_1\delta_{ik}\delta_{jl}t_{mm} + \frac{c_2 t_{ij}t_{kl}}{t_{mm}} = 0 \quad (7b)$$

By making a double contraction on i and j also on k and l i.e. by setting $i = j$ and $k = l$:

$$3c_1 + c_2 = 0 \quad \Rightarrow \quad c_2 = -3c_1 \quad (7c)$$

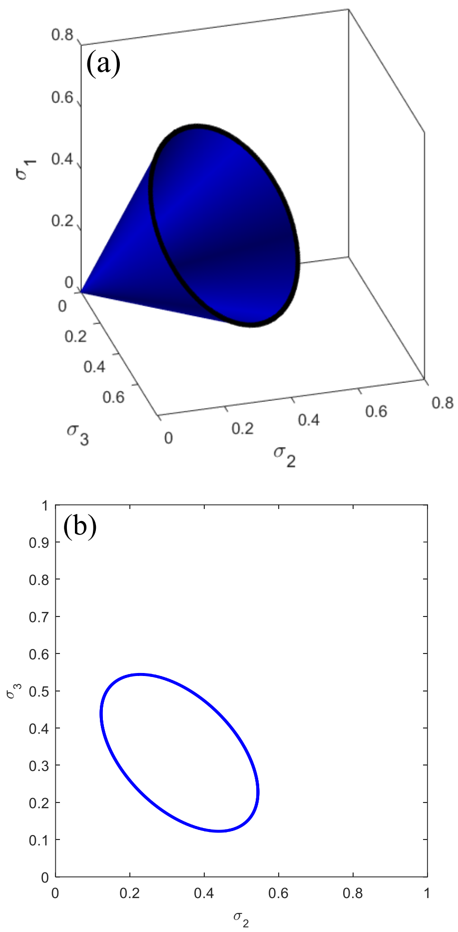

Fig. 2. The limit state surface conical form in (a) the principal stress space and (b) its elliptical projection on the $\sigma_2 - \sigma_3$ plane keeping $I_1 = 1$.

This will reduce the number of independent constants to two making only two tests sufficient for the full prescription of the material constants.

Equivalently, $c_0/c_1 = (c-3)$. By substituting these terms into the hypoelasticity equation, it will take the following form:

$$\boldsymbol{T}^{\blacksquare} = c_1\left[(c-3)(tr\boldsymbol{T})(tr\boldsymbol{D})\boldsymbol{I} + (tr\boldsymbol{T})\boldsymbol{D} - \frac{3tr(\boldsymbol{TD})\boldsymbol{T}}{tr\boldsymbol{T}}\right] \quad (8)$$

4.3. More elaborated dynamical condition

In this part, the condition of letting the material derivative of $\boldsymbol{T}$ vanish instead of its objective rate, is examined and the two procedures are compared. Rationally, this approach is rigor and mathematically correct while the previous one (widely used by other researchers) is correct only when the vorticity tensor, $\boldsymbol{W}$, is zero. Here, we require that $\dot{\boldsymbol{T}} = \boldsymbol{0}$ and the results are compared with the previous one. To make such a comparison, two different objective stress rates, i.e. Jaumann-Zaremba-Noll (Jaumann) and Truesdell rates of Cauchy stress tensor, are employed instead of $\boldsymbol{T}^{\blacksquare}$ with equations summarized in Table 1.

Table 1. Objective stress rates incorporated in the analyses

Row	Objective stress rate, $\boldsymbol{T}^{\blacksquare}$	Equation
1	Jaumann rate	$\boldsymbol{T}^{\circ} = \dot{\boldsymbol{T}} + \boldsymbol{TW} - \boldsymbol{WT}$
2	Truesdell rate	$\boldsymbol{T}^{\square} = \dot{\boldsymbol{T}} - \boldsymbol{LT} - \boldsymbol{TL}^{T} + \boldsymbol{T}\mathrm{tr}\boldsymbol{D}$

4.4. Jaumann rate

By taking the Jaumann rate into account, the form of the limit state will be found as follows:

$$\dot{\boldsymbol{T}} = \boldsymbol{T}^{\circ} - \boldsymbol{TW} + \boldsymbol{WT} = \mathbb{C}^{he}: \boldsymbol{D} - \boldsymbol{TW} + \boldsymbol{WT} = \boldsymbol{0}, \quad \mathbb{C}^{he}_{ijmn} d_{mn} - t_{ik}w_{kj} + w_{ik}t_{kj} = 0 \quad (9a)$$

where $\boldsymbol{W}$ is the vorticity tensor, i.e. $\boldsymbol{W} = \mathrm{skew}\boldsymbol{\nabla v}$ and $\boldsymbol{v}$ is the velocity vector.

To make it possible to take out the common factor associated with the kinematical terms, some manipulations must be made:

$$d_{mn} = \frac{1}{2}(l_{mn} + l_{nm}), \qquad w_{kj} = \frac{1}{2}(l_{kj} - l_{jk}), \quad w_{ik} = \frac{1}{2}(l_{ik} - l_{ki}) \quad (9b)$$

where l_{ij} is the components of the velocity gradient tensor, d_{ij} is the components of the rate of deformation tensor and w_{ij} is the components of the vorticity tensor. Thus:

$$\dot{t}_{ij} = \frac{1}{2}\mathbb{C}^{he}_{ijmn}(l_{mn} + l_{nm}) - \frac{1}{2}t_{ik}(l_{kj} - l_{jk}) + \frac{1}{2}(l_{ik} - l_{ki})t_{kj} \quad (9c)$$

Therefore, after some elaborations:

$$\dot{t}_{ij} = [\mathbb{C}^{he}_{ijmn} + \mathbb{C}^{he}_{ijnm} - t_{ik}\delta_{km}\delta_{jn} + t_{ik}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{kn}t_{kj} - \delta_{km}\delta_{in}t_{kj}] \quad (9d)$$

Since the expression in the parentheses is independent of l_{mn} and also, l_{mn} is arbitrary, then:

$$\mathbb{C}^{he}_{ijmn} + \mathbb{C}^{he}_{ijnm} - t_{ik}\delta_{km}\delta_{jn} + t_{ik}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{kn}t_{kj} - \delta_{km}\delta_{in}t_{kj} = 0 \tag{9e}$$

$$\mathbb{C}^{he}_{ijmn} + \mathbb{C}^{he}_{ijnm} - t_{im}\delta_{jn} + t_{in}\delta_{jm} + \delta_{im}t_{nj} - \delta_{in}t_{mj} = 0 \tag{9f}$$

So far, the expression obtained has been independent of the form of the constitutive equation. For the sake of comparison and clarification, the simple form of the constitutive equation stated earlier is employed and after substitution for this equation, one will get:

$$(3c_0 + 9c_1)t_{mm} + \frac{c_2 t_{ij}t_{ij}}{t_{mm}} = 0 \tag{10}$$

which is exactly the same as what was previously obtained upon setting $\boldsymbol{T}^{\circ} = \boldsymbol{0}$.

4.5. Truesdell rate

By the virtue of incorporation of the Truesdell objective stress rate, one will get:

$$\boldsymbol{T}^{\blacksquare} = \boldsymbol{T}^{\square} = \dot{\boldsymbol{T}} - \boldsymbol{LT} - \boldsymbol{TL}^T + \boldsymbol{T}\mathrm{tr}\boldsymbol{D} \tag{11a}$$

$$\dot{\boldsymbol{T}} = \boldsymbol{T}^{\square} + \boldsymbol{LT} + \boldsymbol{TL}^T - \boldsymbol{T}\mathrm{tr}\boldsymbol{D} = \mathbb{C}^{he}:\boldsymbol{D} + \boldsymbol{LT} + \boldsymbol{TL}^T - \boldsymbol{T}\mathrm{tr}\boldsymbol{D} = \boldsymbol{0} \tag{11b}$$

$$\mathbb{C}^{he}_{ijmn}d_{mn} + l_{im}t_{mj} + t_{im}l_{jm} - t_{ij}d_{kk} = 0 \tag{11c}$$

$$\dot{t}_{ij} = \frac{1}{2}\left[\mathbb{C}^{he}_{ijmn}l_{mn} + \mathbb{C}^{he}_{ijnm}l_{mn}\right] + \left[t_{mj}\delta_{in}l_{nm} + t_{im}\delta_{jn}l_{nm} - t_{ij}\delta_{mn}l_{mn}\right] = 0 \tag{11d}$$

Again, after some mathematical manipulations and further simplifications and substituting the particular form of the constitutive equation for $\mathbb{C}^{he}_{ijmn}$, one will get:

$$(3c_0 + 9c_1 + 3)t_{kk} + \frac{c_2 t_{ij}t_{ij}}{t_{mm}} = 0 \tag{12}$$

This form is clearly different from the one obtained earlier. As a conclusion, the form of the limit state is dependent on the choice of the objective stress rate. More to point, the choice of the objective stress rate affects the material response.

5. The Limit State by the Second Approach

In the second approach, we require the determinant of the matrix representation of the 4th order constitutive tensor, $\mathbb{C}^{he}$, to vanish. To do so, and for the sake of simplicity, the Voigt-Kelvin notation in the principal stress space is employed. One should note that all tensors $\boldsymbol{T}, \boldsymbol{D}$ and $\boldsymbol{T}^{\blacksquare}$ are taken as coaxial, i.e. to have the same eigenvectors and hence, this assumption is valid. In addition, for the Jaumann stress rate, $\boldsymbol{T}^{\blacksquare} = \boldsymbol{0}$ and $\dot{\boldsymbol{T}} = \boldsymbol{0}$ provide the same results (with the proof being a little lengthy) equations are derived for $\boldsymbol{T}^{\blacksquare} = \boldsymbol{0}$. Since the constitutive equation is homogeneous of degree 1 in $\boldsymbol{T}$, one may formally assume that $\mathrm{tr}\boldsymbol{T} = 1$ and hence, the matrix form of this equation by the Voigt-Kelvin notation can be represented as follows:

$$\begin{pmatrix} \sigma_1^{\blacksquare} \\ \sigma_2^{\blacksquare} \\ \sigma_3^{\blacksquare} \\ \sigma_4^{\blacksquare} = 0 \\ \sigma_5^{\blacksquare} = 0 \\ \sigma_6^{\blacksquare} = 0 \end{pmatrix} = \boldsymbol{C}\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 = 0 \\ d_5 = 0 \\ d_6 = 0 \end{pmatrix}, \tag{13a}$$

$$\boldsymbol{C} = \begin{pmatrix} c_2\sigma_1^2 + c_0 + c_1 & c_0 + c_2\sigma_1\sigma_2 & c_0 + c_2\sigma_1\sigma_3 & \boldsymbol{0} \\ c_0 + c_2\sigma_1\sigma_2 & c_2\sigma_2^2 + c_0 + c_1 & c_0 + c_2\sigma_2\sigma_3 & \boldsymbol{0}_{3\times3} \\ c_0 + c_2\sigma_1\sigma_3 & c_0 + c_2\sigma_2\sigma_3 & c_2\sigma_3^2 + c_0 + c_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0}_{3\times3} & \boldsymbol{0} & c_1\boldsymbol{I}_{3\times3} \end{pmatrix}$$

$$\sigma_1 = t_{11}, \quad \sigma_2 = t_{22}, \quad \sigma_3 = t_{33}, \quad \sigma_4 = t_{23}, \quad \sigma_5 = t_{31}, \quad \sigma_6 = t_{12} \tag{13b}$$

$$d_1 = d_{11}, \quad d_2 = d_{22}, \quad d_3 = d_{33}, \quad d_4 = d_{23}, \quad d_5 = d_{31}, \quad d_6 = d_{12} \tag{13c}$$

$$\boldsymbol{0}_{3\times3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{I}_{3\times3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{13d}$$

5.1. Dynamical condition

Again, it is reminded that the limit state, i.e. $\dot{\boldsymbol{T}} = \boldsymbol{0}$ corresponds to the state in which, the kernel of the linear operator defined by $\mathbb{C}^{he}$, becomes non-empty. In other words, there are a subspace of $S(= \mathbb{R}^3 \times \mathbb{R}^3)$ where the stress rate vanishes for a series of motions. Therefore, it can be immediately concluded that the limit state is a necessary condition for vanishing the rate of the stress tensor and hence, there may be stress paths which can escape from the limit state. Now, we derive this necessary condition by setting the determinant of $\boldsymbol{C}$ to zero. An important question is whether the determinant of $\boldsymbol{C}$ (like the one for an arbitrary 2nd order tensor) is invariant. Of course, it is not and it may rise serious questions on the validity of this approach. However, it is possible to mathematically verify whether the determinant of $\boldsymbol{C} = 0$ is zero in all coordinate systems where all necessary tensors are defined (Private communication with Prof. Mojtaba Mahzoon, Professor Emeritus in Applied Mechanics and Mathematics, School of Mechanical Engineering, Shiraz University, Shiraz, Iran, Winter 2023). Now, setting the determinant of $\boldsymbol{C}$ to zero, with a rather lengthy operations eventually gives the following results:

$$\det \boldsymbol{C} = c_1^6(c_1^3 + c_3 c_1^2\sigma_1^2 + c_3 c_1^2\sigma_2^2 + c_3 c_1^2\sigma_3^2 + 3c_0 c_1^2 + 2c_0 c_3 c_1 \sigma_1^2 - 2\, c_0 c_3 c_1 \sigma_1 \sigma_2 - 2c_0 c_3 c_1 \sigma_1 \sigma_3 + 2c_0 c_3 c_1 \sigma_2^2 - 2c_0 c_3 c_1 \sigma_2 \sigma_3 + 2c_0 c_3 c_1 \sigma_3^2) = 0 \quad (14)$$

By incorporating the assumption that $\text{tr}\boldsymbol{T} = 1$, one will get the form of the limit state as follows:

$$\text{tr}\boldsymbol{T}^2 + c(\text{tr}\boldsymbol{T})^2 = 0 \quad \text{or} \quad (1+c)I_1^2 - 2I_2 = 0 \quad (15a)$$

$$c = \frac{(c_1^2 + 3c_0 c_1 - c_0 c_3)}{(c_3 c_1 + 3c_3 c_0)} \quad (15b)$$

$$J_2 + \kappa^2 I_1^2 = 0, \quad \kappa^2 = -\frac{(1+3c)}{6}, \quad c < -\frac{1}{3} \quad (15c)$$

This form is exactly the same as the one obtained earlier, but with different coefficients.

So far, the main goal of this research has been achieved, i.e. to show that the equations of hypoelasticity are capable of providing the limit state and the form of the limit state may be similar to those obtained in the classical theory of plasticity, in particular, the Drucker-Prager criterion. Two approaches were taken setting $\boldsymbol{T}^{\blacksquare} = \boldsymbol{0}$ and by setting the determinant of the coefficient matrix (corresponding to the 4th order constitutive tensor) to zero. While both approaches may qualitatively show similar results, the reason why the equation just derived, results in a more practical range for the limit state will be elucidated.

6. Application

Despite the main goal being conducting a rational study and providing some analytical equations, an application of the results thus obtained is presented in predicting the material behavior for a particular type of pressure-sensitive materials. To do so, we employ the dataset of a pressure-sensitive material which in this case is the Quartz Sand, from dunes at Kurnell, Sydney, Australia (experimental data from [23]) in the standard triaxial compression test, i.e. with a symmetrical loading with $\sigma_2 = \sigma_3$ and $d_2 = d_3$. Sand properties and constants of the constitutive equation are presented in Table 2. Fig. 3 shows the material response and the stress paths taken in three tests at different confining pressures, $p\ (= I_1/3)$, towards the limit state. Finite deformation as well as the Jaumann stress rate have been assumed. It is clear that the infinitesimal deformation theory cannot be accurately applied as strains are far beyond the range of infinitesimal strains.

Table 2. Parameters of the Quartz Sand [23]

Quartz Sand (data from [23])	
Parameter	**Value**
Mean grain size (mm), D_{50}	0.31
Coefficient of uniformity, C_u	1.83
Specific gravity, G_s	2.65
Minimum void ratio, e_{min}	0.60
Maximum void ratio, e_{max}	0.92
Constitutive equation constants (non-dimensional)	
Constant	**Value**
c_0	-10.0
c_1	-8.0
c_2	+40

7. Discussion

In this study, by assuming a particular form of a hypoelastic constitutive equation and a generally applicable procedure (by two different approaches), an equation for the limit state equation, based on the parameters of the constitutive equation that involves only one parameter, κ, resembling the Drucker-Prager yield criterion was obtained. This parameter was found to be $\kappa = 1.05$ and $\kappa = 0.26$ when examined against some experimental data, corresponding to the first and

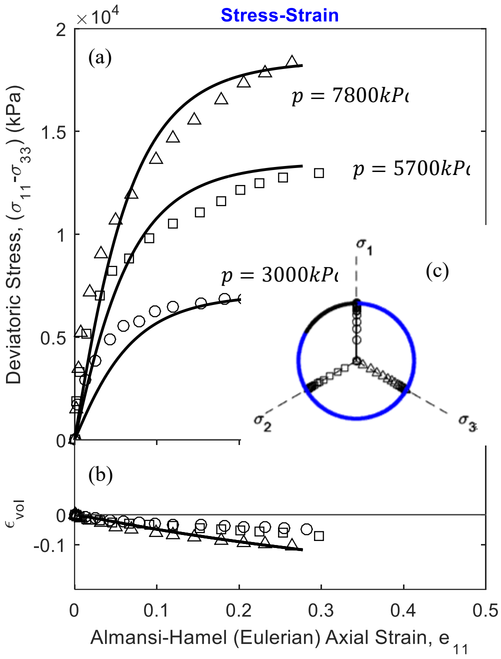

Fig. 3. Modeling of the Quartz Sand: (a) stress-strain response, (b) volume change and (c) stress paths in the tests on the Haigh-Westergaard plane towards the limit state.

second approaches, respectively. Therefore, while the first approach, i.e. requiring the constitutive tensor to vanish, gives no information on the limit state, the second approach provides meaningful results corresponding to a typical value of κ analogous to the one obtained by classical plasticity theory and Drucker-Prager failure criterion. In other words, the first value corresponds to a conical region which embodies the entire stress space explaining the absence of information on the limit state. Therefore, it seems that the second approach is more suitable in the case of linear kernels as it defines a limited region in the stress space, whereas the first approach seems to cover the entire space, a result that is supported by experimental data. Again, this limit state defines a bound which necessitates the stress paths to approach the limiting state, but it does not guarantee the non-escaping stress paths from this boundary. This means that the limit state thus obtained is only a necessary condition for stress paths to reach such a state and not a sufficient condition.

In addition to modeling pressure-sensitive materials, like porous and granular materials, metallic (often pressure insensitive materials) can be successfully modeled by the same constitutive equation. To do so, it is only required to slightly change the structure of the equation to achieve an equation resulting in a limit state independent of the hydrostatic pressure. Such an equation for the limit state resembles the von Mises yield/failure criterion and can, hence, be successfully applied to metals. This is an open research area and is beyond the length of this paper.

8. Conclusion

The primary goal of this paper was to study the possibility of utilizing hypoelasticity in predicting the limit state. In this regard, three secondary objectives were delineated: (a) the effect of the choice of the objective stress rate, (b) the requirements of the limit state, i.e. setting the constitutive tensor equal to zero or requiring the kernel of the linear operator to be non-empty and (c) the comparison of the two approaches. Despite the rationale form of all equations and general derivations, disrespectful of the form of the constitutive equation, a simple constitutive equation, i.e. a homogeneous of degree one equation in the stress tensor, has been suggested and derivations were performed for this equation. Results indicated that:

(a) the limit state can be successfully predicted by the hypoelastic constitutive equations, and the forms of equations are analogous to the forms of traditional failure/yield criteria of classical theory of plasticity.

(b) The effect of the objective stress rate is very important, both in the material response and the limit state. This research specifically made comparisons between the general form of the limit state equation for two objective stress rates, i.e. Jaumann-Zaremba-Noll and Truesdell. It was found that both rates give a similar form of the limit state equation but with different coefficients.

(c) While both approaches outlined in this research may provide the same form of the limit state equation, it appears that the second approach, i.e. a non-empty kernel for the linear operator of the hypoelastic constitutive equation, seems to give rise to a practically better boundary for the limit state. It can be observed by looking into the results obtained by the calibration of an experimental data. The reason lies in the boundary defined by the limit state which, only in the second approach covers a limited region in the stress space.

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Conflict of Interest

There is no conflict of interest.

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9. References

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